HSJOM Competition Winter Solutions

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Winter 2019

- This is a 25-question, multiple choice test. Each question is followed by answers marked A, B, C, D and E. Only one of these is correct.
- You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
- No aids are permitted other than scratch paper, graph paper, a ruler, a compass, a protractor and erasers. Calculators are not allowed on the test. No problems on the test will *require* the use of a calculator.
- Figures are not necessarily drawn to scale.
- You will have **100 minutes** to complete the test.
- To submit your answers, use the **google form** sent to your DMs. A leaderboard will be made starting at the conclusion of the test; if you wish to remain anonymous, please state so when submitting your answers.
- After the contest, a discussion board will open for users to submit their solutions and vote on others. The most creative and interesting approaches will be put on our official solutions sheet!
- Problems are categorized as follows:
  - (ALG - Algebra)
  - (NT - Number Theory)
  - (C&P - Counting and Probability)
  - (GEO - Geometry)
  - (ADV - Advanced Mathematics)
  - (MISC - Miscellaneous)

1. The American Mathematics Competition series is a competition series often abbreviated “AMC.” Like the AMC, the HSJOM Competition has a similar scoring system involving 6 points per correct answer, 1.5 per omitted answer, and 0 per incorrect answer. What is the largest score under 150 that can be obtained? (ALG)

   A. 100 B. 148.5 C. 144 D. 145.5 E. 141

2. Josh M. and J. Ohms are on opposite ends of a bridge 2 miles long. If they run towards each other with constant speed, how many times faster is J. Ohms than Josh M. if Josh M. travels $\frac{2}{3}$ of a mile before meeting J. Ohms? (ALG)

   A. $\frac{1}{3}$ B. $\frac{1}{2}$ C. 1 D. 2 E. 4

3. A positive integer is **introverted** if it is not divisible by its units digit. For example, 249 is introverted since $9 \nmid 249$. Compute the number of three-digit introverted numbers. (NT)

   A. 140 B. 200 C. 255 D. 420 E. 480

4. The Art of Problem Solving (AoPS) hosts an online community of students passionate about mathematics. A curious student, Josh M., decided to randomly change the case and order of the string "AoPS." For example, "aoPs" and "PsAo" are both orderings that can be achieved. How many unique orderings can be achieved (including "AoPS")? (C&P)

   A. 16 B. 24 C. 96 D. 384 E. 1440

5. What is the angle between the hour hand and the minute hand on an analog clock when the time is $5:20$? (Misc)

   A. $25°$ B. $30°$ C. $35°$ D. $40°$ E. $45°$

6. The **modulus** of a complex number is its distance from the origin. Let x and y be (not necessarily distinct) real numbers on the interval $[0, 8]$. What is the probability that the modulus of the complex number $x + yi$ (where $i = \sqrt{-1}$) is less than 6? (C& P)

   A. $\frac{3}{32}$ B. $\frac{1}{4}$ C. $\frac{\pi}{8}$ D. $\frac{9\pi}{64}$ E. $\frac{3\pi}{16}$

7. Mathematics is an inherently collaborative academic pursuit. Most problems are solved not through a single person’s effort but instead by a collective attempt from the community. Josh M. and his acquaintance, J. Ohms, along with 8 other friends are deciding how to split into two groups of 5 to tackle two separate mathematics problems. Given that he must be separate from J. Ohms how many different groupings are possible? (C&P)

   A. 24 B. 35 C. 70 D. 1680 E. 40320

8. The **floor function** is a function that takes a real number and outputs the greatest integer less than or equal to the real number. It is denoted by $\lfloor k \rfloor$ (e.g. $\lfloor 6.7 \rfloor = 6$). How many positive integers n are there such that $\lfloor \frac{n^2}{5} \rfloor$ is prime? (NT)

   A. 0 B. 1 C. 2 D. 3 E. Infinitely Many

9. If $x = 15°$ and $y = 30°$, compute $(1 + \tan x)(1 + \tan y)$. (ALG)

   A. $\sqrt{2} - 1$ B. $\frac{\sqrt{3}}{2}$ C. 1 D. $\sqrt{2}$ E. 2

10. A **base** in mathematics represents the number of digits a system uses to represent numbers. For example, the binary system uses base 2, meaning all numbers can only be represented with some combination of 1 or 0. Bases are denoted with subscripts. Thus, the number $8_{10} = 1 * 2^3 + 0 * 2^2 + 0 * 2^1 + 0 * 2^0 = 1000_2$.

    The number 2019 in base 2 is written as 11111100011_2. A **swap** consists of choosing any "1" and swapping the 1 and the number to it's right to the opposite digit (ignoring the empty spot to it's right if the 1 is in the units digit place). (e.g. swapping on $10 \to 01$). What is the maximum number of swaps that can be performed on 2019 in base 2? (NT)

    A. 4 B. 29 C. 45 D. 53 E. 54

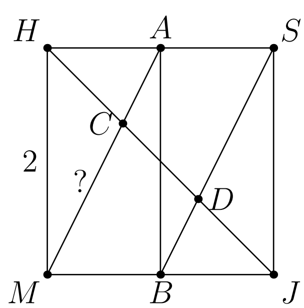
11. There is, in mathematics, a number that is not a real number called an **imaginary number**. Imaginary numbers are denoted by i where $i = \sqrt{-1}$. How many pairs of real numbers (a, b) are there such that $(a + bi)^7 = a - bi$. (ALG)

    A. 1 B. 3 C. 7 D. 8 E. 9

12. A unit fraction $\frac{1}{n}$ is considered *cool* if the prime factorization of n consists of single-digit primes. What is the sum of all cool fractions? (NT)

    A. $\frac{27}{8}$ B. $\frac{61}{35}$ C. $\frac{96}{35}$ D. $\frac{19}{5}$ E. $\frac{41}{14}$

13. A common trick in origami to fold a paper in thirds involves the following partitions of a square $HSJM$. Fold the square paper in half from the center A of side $\overline{HS}$ to the center B of the opposite side $\overline{JM}$. Then, unfold the paper and fold it in half again along diagonal $\overline{HJ}$. Finally, form the line $\overline{MA}$ and the line $\overline{BS}$ and mark the intersections of both lines with diagonal $\overline{HJ}$ as C and D respectively. Parallel lines to either set of sides of the square crossing through these two points will split the square into thirds.



    Suppose $HSJM$ has a side length of 2. What is the length of segment $\overline{MC}$ as shown in the diagram above? (GEO)

    A. 1 B. $\sqrt{2}$ C. $\frac{2\sqrt{2}}{3}$ D. $\frac{2\sqrt{5}}{3}$ E. $\frac{3\sqrt{5}}{4}$

14. Find the number of ordered pairs (x, y), where x and y are distinct positive integers, which satisfy
$$\frac{x+5y}{x-y} > xy.$$
(NT)

A. 3 B. 4 C. 5 D. 6 E. 7

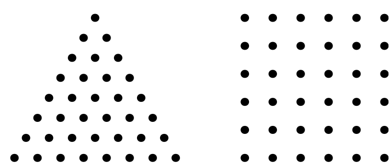
15. The factorial function "!" (denoted on positive integers as $n!$) results in the product of all integers $n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$. If $17! = 355xy7428096000$, compute the product of x and y. (NT)

A. 48 B. 56 C. 30 D. 15 E. 42

16. Josh M. was watching his favorite youtuber, Sal Khan, when his acquaintance, J. Ohms, offered him a plate of 8 cookies. Unfortunately, only one is real and weighs heavier than the rest (which all weight the same)! All Josh M. has to find the real cookie is a double-pan balance scale. What is the minimum number of comparisons Josh needs to make to find the real cookie? (MISC)

A. 1 B. 2 C. 3 D. 4 E. 7

17. A positive integer k is *slick* if k points can be rearranged to form a square and a triangle. The two smallest *slick* integers are 1 and 36. Find the remainder when the absolute difference of the next two smallest *slick* integers are divided by 100. (NT)



A. 4 B. 24 C. 51 D. 79 E. 91

18. Ten chairs are set up in a row for an HSJOM pie-eating competition. Only five contestants attend the competition, but two contestants, Josh M. and J. Ohms, refuse to sit next to each other. In how many ways can the contestants be seated? (C&P)

A. 24192 B. 25124 C. 6048 D. 18026 E. 20920

19. Recently, Josh M. received a mysterious package of $(1+7+7^2+7^3+7^4+7^5+7^6+7^7)$ resistors in the mail. Suspecting the package was from his acquaintance, J. Ohms, Josh M. wishes to divide the resistors into a prime number of piles each with the same number of resistors to sort through the strange gift. What is the largest number of piles Josh M. can create with no resistors left over? (ALG)

A. 5 B. 47 C. 727 D. 1201 E. 2017

20. Consider acute angles $\angle A, \angle B$ which satisfy:
$$\sin^2 A(2 - 2\cos B) = \sin^2 B$$
If $\angle A = 54°$, find $\angle B$. (ALG)

A. $9°$ B. $18°$ C. $36°$ D. $60°$ E. $72°$

21. Evaluate the sum $\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\frac{6^a-a^2b+b^2a}{6^{2a+b}+6^{2b+a}}$. (ALG)

A. $\frac{2}{3}$ B. $\frac{18}{25}$ C. $\frac{29}{36}$ D. $\frac{43}{25}$ E. 2

22. While touring Paris, France (the birthplace of the renowned L'Hôpital), Josh M. came across a strange machine that outputs 8 Francs for every 2 dollars he put in, 27 Francs for every 3 dollars he put in, 119 Francs for every 5 dollars he put in, and 216 Francs for every 6 dollars he put in. On the machine, a sticker indicates that the conversion is based off of a third degree polynomial with leading coefficient 2. Intrigued, Josh decides to put in 4 dollars into the machine. How many Francs does Josh receive? (ALG)

A. 30 B. 32 C. 60 D. 64 E. 108

23. Let $\triangle ABC$ be a triangle with $AB = 5, AC = 6$, $BC = 7$, and circumcircle Ω. The two circles that are tangent to AB, AC, and Ω touch Ω at points P and Q respectively. If PQ intersects BC at T, find AT. (GEO)

A. $\frac{120}{7}$ B. $\frac{315}{17}$ C. $\frac{210}{11}$ D. $\frac{96}{5}$ E. $\frac{343}{17}$

24. If the sequence $\{a_n\}$ is recursively defined by $a_1 = 2$, and $a_{n+1} = a_n + \frac{n(n+1)}{2}$ for $n \geq 1$, calculate the units digit of a_{100}. (NT)

A. 2 B. 4 C. 5 D. 6 E. 9

25. Consider another $\triangle ABC$, with $AB = 5$, $AC = 6$ and $BC = 7$. The incircle of $\triangle ABC$ touches AC at X and BC at Y. AX and BY intersect at P. Find the area of $\triangle APB$. (GEO)

A. $\frac{16\sqrt{6}}{11}$ B. $\frac{18\sqrt{6}}{11}$ C. $\frac{12\sqrt{6}}{13}$ D. $\frac{18\sqrt{6}}{13}$ E. $\frac{24\sqrt{6}}{13}$

1. Because the max score is 150 on the AMC, the next possible max score occurs when you get 24 correct and omit 1. This is because you must have one question that is not correct, and omitting a question gives more points than getting it wrong. Thus, the answer is $\boxed{\textbf{D. 145.5}}$

2. Suppose that they both travel for t minutes. Then Josh M.'s speed is $\frac{2}{3t}$ miles per minute and J. Ohms' speed is $\frac{4}{3t}$ miles per minute. Thus the ratio of their speeds is $\frac{4}{3t} \div \frac{2}{3t} = \boxed{\textbf{D. 2}}$.

3. The problem can be bashed with casework. The number of three digit numbers that aren't introverted is 420, so the number of three digit introverted numbers is $900 - 420 = \boxed{\textbf{E. 480}}$.

4. Ignoring the capital cases for now (we'll deal with it later), there are 4! different orderings for the four unique letters. Then, we multiply this by 2^4 because of the upper and lower cases for the four letters. So, our answer is $4! * 2^4 = \boxed{\textbf{D. 384}}$

5. There are 360° in a circle and 12 hours on a clock, so each hour corresponds to 30°. At 5:20, the hour hand is one-third of an hour past 5, so the angle between the hour hand and "5" is $\frac{1}{3} \cdot 30° = 10°$. The minute hand points to "4", so the angle between the minute hand and our hand is 30°. We sum the two angles up to find the desired angle of $30° + 10° = \boxed{\textbf{D. } 40°}$.

6. For the modulus of the complex number $x + yi$ to be less than 3,
$$x^2 + y^2 < 9$$
As the total area that can contain the point is $8^2 = 64$, and the desired region is a circle with area 9π, the probability that the point is in the circle is $\boxed{\textbf{D. } \frac{9\pi}{64}}$.

7. Because the two groups are indistinct, begin by arbitrarily placing Josh M. into group 1 and J. Ohms into group 2. Then, the total number of orderings is now just filling the four spots on either team from the 8 available people. So, $\binom{8}{4} = \boxed{\textbf{C. 70}}$

8. Suppose that $n = 5a + b$ where a, b are integers with $0 \le b \le 4$. If $b = 0$, then $\lfloor \frac{n^2}{5} \rfloor = 5a^2$ which is prime only when $a = 1$. If $b = 1$, then $\lfloor \frac{n^2}{5} \rfloor = 5a^2 + 2a = a(5a+2)$ which is prime only when $a = 1$. If $b = 2$, then $\lfloor \frac{n^2}{5} \rfloor = 5a^2 + 4a = a(5a + 4)$ which is never prime. If $b = 3$, then $\lfloor \frac{n^2}{5} \rfloor = 5a^2 + 6a + 1 = (5a + 1)(a + 1)$ which is never prime. Finally, if $b = 4$, then $\lfloor \frac{n^2}{5} \rfloor = 5a^2 + 8a + 3 = (5a + 3)(a + 1)$ which is prime only when $a = 0$. Thus the only primes are when $n = 4, 5, 6$ so $\boxed{\textbf{D. 3}}$.

9. Note that
$$1 = \frac{\tan(15°) + \tan(30°)}{1 - \tan(15°)\tan(30°)}$$
$$\implies 1 - \tan(15°)\tan(30°) = \tan(15°) + \tan(30°)$$
$$\implies 1 + \tan(15°) + \tan(30°) + \tan(15°)\tan(30°) = (1 + \tan(15°))(1 + \tan(30°)) = \boxed{\textbf{E. 2}}$$

10. Observe, that in order to achieve the largest number of swaps, we need to always choose to swap on the right-most one to avoid eliminating any other ones. Thus, starting from right to left, we see that the number of swaps to fully eliminate that one is simply n where the one is located in the nth position. 2019 in binary has 1s in the $\{1, 2, 6, 7, 8, 9, 10, 11\}$ positions. Summing, we have $1 + 2 + 6 + 7 + 8 + 9 + 10 + 11 = \boxed{\textbf{E. 54}}$

11. Let $z = a + bi$. Then we are given that $z^7 = \overline{z}$, the conjugate of z. Since $|\overline{z}| = |z|$, we have that $|z|^7 = |z|$ so $|z| = 1$ or $|z| = 0$. If $|z| = 0$, then $z = 0 + 0i$ is the only solution. If $|z| = 1$, then $z^7 = \overline{z} = \frac{1}{z}$ so $z^8 = 1$. Thus z can be any of the eighth roots of unity, giving us 8 more solutions. Therefore the answer is $\boxed{\textbf{E. 9}}$

12. The series of cool fractions is generated by the expansion of

$$\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots\right)\left(1 + \frac{1}{3} + \frac{1}{9} + \cdots\right)\left(1 + \frac{1}{5} + \frac{1}{25} + \cdots\right)\left(1 + \frac{1}{7} + \frac{1}{49} + \cdots\right)$$

Via the formula for the sum of an infinite geometric sequence, the product above is $\frac{35}{8}$. However, $\frac{1}{1}$ is not a cool fraction, which makes the answer $\frac{35}{8} - 1 = \boxed{\textbf{A. } \dfrac{27}{8}}$

13. First, notice that perpendiculars fromm C and D to all sides trisect both the vertical and horizontal sides of the squares. Since C lies on $\overline{AM}$ and $\overline{AM}$ touches both the bottom and top side, C trisects $\overline{AM}$. Finally, the length of $\overline{CM}$ is simply two-thirds of the length of $\overline{AM}$, or $\overline{CM} = \frac{2}{3} * \overline{AM} = \frac{2}{3} * \sqrt{1^2 + 2^2} = \frac{2}{3} * \sqrt{5} = \boxed{\textbf{D. } \frac{2\sqrt{5}}{3}}$

14. Clearly, $x > y$. Suppose that (x, y) is a solution. Then, $(x - 1, y)$ is also a solution, provided that $x - 1 \neq y$, since LHS increases while RHS decreases. This also means that if (x, y) is not a solution, then neither is $(x + 1, y)$.
Suppose that we have $(y + a, a)$ as a solution.
Case: $a = 1$. Then, $1 + 5y - y^2 > 0$, from which we obtain $y = 1, 2, 3, 4$ or 5.
Case: $a = 2$. Then, $1 + y - y^2 > 0$, from which we obtain $y = 1$.
Case: $a = 3$. Then, $1 - y - y^2 > 0$, which does not have solutions.
By our previous claim, we know that there are no more solutions. Thus, the number of solutions is $\boxed{\textbf{D. 6}}$

15. Note that via divisibility rules, $x + y \equiv 5 \pmod 9$ and $y - x \equiv 2 \pmod{11}$. The only possible case we have is $x + y = 14$ and $y - x = 2$. Solving the system, we get $(x, y) = (6, 8)$, which makes $x \cdot y = \boxed{\textbf{A. 48}}$.

16. First, it is trivial to see that one weighing is impossible to always determine the fake coin. However, two weighings can determine the fake coin as follows:

    (a) Take any 6 coins and weigh three against three. If the balance is even, continue to "(b)." Otherwise, continue to "(c)."

    (b) Since the coins all balance, they must all be real else the balance would be off. Thus, all that remains is to test the last two coins, and the one on the higher balance is the fake.

(c) Because the fake coin weighs less than the real ones, the higher side of the balance contains the fake coin. There are three coins that are possible candidates for the fake coin. Choose two coins at random and weigh them against each other. Now we solve by casework:

i. If the pan balance is not balanced, then the fake is among the two coins chosen, and the coin on the higher side of the balance is the fake.

ii. If the pan is balanced, then the fake must be the third coin that was not weighed.

Thus, our answer is $\boxed{\textbf{B. 2}}$

17. A positive integer k is *slick* if it is both a perfect square and a triangular number, i.e. $k = a^2 = \frac{b(b+1)}{2}$ for some integers a, b. Since b and $b+1$ are relatively prime, if b is even, then $b/2$ and $b+1$ must both be perfect squares, and if b is odd, then $(b+1)/2$ and b must be perfect squares. Listing perfect squares, we find that $b = 49$ and $b = 288$ are the smallest such numbers that work. Thus $k = 49 \cdot 25$ and $k = 144 \cdot 289$ and we compute their difference modulo 100 as $\boxed{\textbf{E. 91}}$

18. There are $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6$ total combinations. Suppose Josh and J. Ohms are sitting next to each other: there are $2 \cdot 9 \cdot 8 \cdot 7 \cdot 6$ ways for this to happen. Via complementary counting, the total number of combinations is $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 - 2 \cdot 9 \cdot 8 \cdot 7 \cdot 6 = \boxed{\textbf{A. 24192}}$

19. $1 + 7 + 7^2 + 7^3 + 7^4 + 7^5 + 7^6 + 7^7 = \frac{7^8-1}{7-1}$. Since we have the 6 in the denominator, unless 3 or 2 is the largest prime factor of $7^8 - 1$ our answer will not be affected (and by observation of the answer choices our result won't be affected anyways). Thus, our answer is just the largest prime factor of $7^8 - 1$.

$7^8 - 1 = (7^4+1)(7^4-1) = (7^4+1)(7^2-1)(7^2+1) = 2402 \cdot 48 \cdot 50 = (2 \cdot 1201) \cdot (2^4 \cdot 3) \cdot (2 \cdot 5^2)$

By checking through all primes from 2 to $\sqrt{1201}$, we see that 1201 is indeed prime. Thus, our answer is $\boxed{\textbf{D. 1201}}$

20. Recall that $\sin^2 \frac{B}{2} = \frac{1-\cos B}{2}$ and $\sin B = 2\cos\frac{B}{2}\sin\frac{B}{2}$, so we have

$$\sin^2 A \cdot 4\sin^2 \frac{B}{2} = 4\sin^2\frac{B}{2}\cos^2\frac{B}{2}$$

Since $\sin^2 \frac{B}{2} \neq 0$, we get $\sin^2 A = \cos^2 \frac{B}{2}$. Also, $B < 90°$, so $90° - A = \frac{B}{2}$. Thus, our answer is $\boxed{\textbf{E.}\ 72°}$

21. Suppose that

$$X = \sum_{a=0}^{\infty}\sum_{b=0}^{\infty} \frac{6^a - a^2b + b^2a}{6^{2a+b} + 6^{2b+a}}.$$

Then by symmetry,

$$X = \sum_{a=0}^{\infty}\sum_{b=0}^{\infty} \frac{6^b - b^2a + a^2b}{6^{2a+b} + 6^{2b+a}}.$$

Adding them together,

$$2X = \sum_{a=0}^{\infty}\sum_{b=0}^{\infty} \frac{6^a+6^b}{6^{a+b}(6^a+6^b)} = \sum_{a=0}^{\infty}\sum_{b=0}^{\infty} \frac{1}{6^{a+b}}.$$

This expression is equal to $\left(\sum_{i=0}^{\infty} \frac{1}{6^i}\right)^2 = \left(\frac{6}{5}\right)^2 = \frac{36}{25}$ so $X = \frac{18}{25}$. Thus the answer is $\boxed{\textbf{B. } \frac{18}{25}}$.

22. Notice that the question provides us with four cases. Since four cases is enough to generate a system of four equations with four unknowns, the machine will be able to be expressed as a cubic function. Additionally, notice how three inputs are simply the cube of the input. With some clever choice of terms, the strange machine Josh M. has come across can be represented as the following function: $f(x) = x^3 + (x-2)(x-3)(x-6)$. Checking to see if 5 dollars satisfies the function, we see that indeed: $f(5) = 5^3 + (5-2)(5-3)(5-6) = 125 + (3)(2)(-1) = 119$. So when Josh inputs 4 dollars, he will receive $f(4) = 4^3 + (4-2)(4-3)(4-6) = 64 + (2)(1)(-2) = 64 - 4 = \boxed{\textbf{C. 60}}$ Francs.

23. We use a transformation Ψ known as $\sqrt{bc}$-inversion, consisting of an inversion centered at A with radius $\sqrt{AB \cdot AC}$ followed by a reflection about the angle bisector of $\angle A$. Notice that Ψ swaps B and C and swaps (ABC) with BC.

 Since lines AB and AC swap, the two circles tangent to AB, AC, and (ABC) must map to the incircle and the A-excircle of $\triangle ABC$. Therefore, P and Q map to the points at which the incircle and A-excircle are tangent to BC, which we define as P' and Q' respectively. Finally, T maps to T', the intersection of (ABC) and $(AP'Q')$. I claim that this point is the reflection of A across the perpendicular bisector of BC. To show that this point lies on $(AP'Q')$, it suffices to show that $BP' = CQ'$, which is true since $BP' = CQ' = s - b$ where s is the semiperimeter $\frac{1}{2}(AB + BC + AC)$ and $b = AC$.

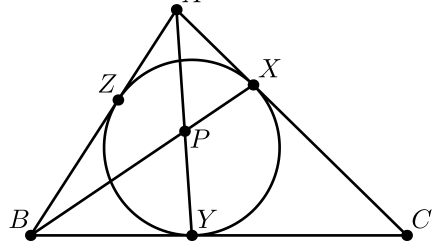
 Now if D is the foot of the altitude from A to BC, we have that

$$AT' = BC - 2 \cdot BD = 7 - 2 \cdot \sqrt{25 - AD^2}.$$

 With Heron's formula, we compute $AD = \frac{12\sqrt{6}}{7}$ so $AT' = \frac{11}{7}$. Since $AT \cdot AT' = AB \cdot AC = 30$, we have $AT = \frac{210}{11}$ or $\boxed{\textbf{C. } \frac{210}{11}}$.

24. Note that the units digit of a_n repeats in cycles of twenty. Thus,

$$a_{20} \equiv a_{100} \pmod{10} = \boxed{\textbf{A. 2}}$$

25. 

It is well known that the two tangents from a point to a circle have equal lengths. Thus, $AX = AZ$, and likewise with B, C. Letting $AX = AZ = l_a$, and similar definitions for l_b, l_c, we have other $l_a + l_b = c$ and cyclic variations. From this, we find that $l_a = 2, l_b = 3, l_c = 4$.
We also know that

$$[ABP] = \frac{BP}{BX}[ABX] = \frac{BP}{BX} \cdot \frac{AX}{AC}[ABC]$$

By using mass points (other similar techniques also work), we are able to find out $\frac{BP}{BX}$. Assign mass 6 to A, so mass B and C recieves 4 and 3, respectively. Then, X has mass 9. So, $BP : XP = 9 : 4$, and $\frac{BP}{BX} = \frac{9}{13}$. Also, $\frac{AX}{AC} = \frac{1}{3}$.
Finally, we need to find $[ABC]$. This is achievable with Heron's formula, which gives $[ABC] = 6\sqrt{6}$. The final answer is then $\boxed{\textbf{D. } \frac{18\sqrt{6}}{13}}$